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Abstract

This paper concerns the study of the asymptotic behavior of solutions to reaction-diffusion systems modelling multi-components reversible chemistry with spatial diffusion. By solution, we understand any limit of adequate approximate solutions. It is proved in any space dimension that, as time tends to infinity, the solution converges exponentially to the unique homogeneous stationary solution. We adapt and extend to any number of components, the entropy decay estimates which have been exploited for some particular 3×3 and 4×4 systems.

1 Introduction

The purpose of the present paper is to describe the *asymptotic behavior* as time tends to infinity of the solutions to reaction-diffusion systems arising in the modelization of reversible chemical reaction with multi-components $\{A_i\}_{1\leq i\leq N}$

$$\alpha_1 A_1 + \cdots + \alpha_m A_m \rightleftharpoons \alpha_{m+1} A_{m+1} + \cdots + \alpha_N A_N \tag{1}$$

where $m, N, \alpha_k, k = 1, ..., N$ are positive integers with $1 \leq m < N$.

Let $u_k = u_k(x,t)$ be the concentration of A_k at position $x \in \Omega \subset \mathbb{R}^n$ and time $t \in [0,T), T > 0$ (Ω will be assumed to be open, bounded and with a regular boundary throughout the paper). According to the mass action law (with reaction rates c_1 from left to right and c_2 from right to left) and according to Fick's law for the diffusion, the evolution of $u = (u_1, ..., u_N)$ is described by the reaction-diffusion system

$$\left\{ \begin{array}{l} \frac{\partial u_k}{\partial t} - d_k \Delta u_k = \chi_k f(u) \quad \text{in } Q_T = \Omega \times (0,T), \\ \frac{\partial u_k}{\partial \nu}\big|_{\partial\Omega} = 0, \quad u_k\big|_{t=0} = u_{k0}(x) \geq 0, \quad 1 \leq k \leq N, \end{array} \right. \tag{2}$$

where $d_k > 0$, $1 \leq k \leq N$, ν is the outer unit normal vector and

$$f(u) = c_1 \prod_{j=1}^{m} u_j^{\alpha_j} - c_2 \prod_{j=m+1}^{N} u_j^{\alpha_j}, \quad \chi_k = \left\{ \begin{array}{ll} -\alpha_k, & 1 \leq k \leq m \\ \alpha_k, & m+1 \leq k \leq N. \end{array} \right. \tag{3}$$

We prove in this paper that *"global solutions"* on $[0, \infty)$ of (2) converge exponentially in $L^1(\Omega)$ as $t \to +\infty$ to a well-defined (and unique) homogeneous stationary solution of System (2) (see Theorem 3 for a precise statement). As explained below, this extends to the general situation (2) similar results obtained in case of 3×3 or 4×4 systems [5, 6, 7, 9].

In order to state precisely our asymptotic result (see Theorem 3), let us first recall what is known about the rather difficult question of *global existence in time* of solutions to (2). Note for instance, that it is not yet understood in dimension $n \geq 3$ and for general diffusion coefficients $d_k \in (0, \infty)$, whether global classical solutions exist for the model quadratic case $m = 2, N = 4, \alpha_k = 1$, that is $f(u) = c_1 u_1 u_2 - c_2 u_3 u_4$!

Global classical solutions do exist for this f in space dimension $n = 1, 2$ (see e.g. [11, 16, 2]). More generally, global existence is also proved for (2) when the space-dimension n is small enough with respect to the degree of the polynomial f or when the diffusion coefficients d_k are close enough to each other (see the discussion in [14]).

But let us recall what the situation is for a general space-dimension n and general positive $d_k \in (0, \infty)$ (we assume $c_1 = c_2 = 1$ for simplicity).

1. If $m = 1, N = 2$ (that is $f(u) = u_1^{\alpha_1} - u_1^{\alpha_2}$), then global existence of uniformly bounded (and therefore classical) solutions easily follows from the invariance of the rectangles

   $$\{(u_1, u_2); 0 \leq u_1 \leq M_1, 0 \leq u_2 \leq M_2\} \text{ where } M_1^{\alpha_1} = M_2^{\alpha_2}.$$

2. If $N = m + 1, \alpha_N = 1$ (i.e. $f(u) = \prod_{k=1}^{m} u_k^{\alpha_k} - u_N$), then global classical solutions do also exist (see [1]). The same symmetrically holds if $m = 1, \alpha_1 = 1$ ($f(u) = u_1 - \prod_{k=2}^{N} u_k^{\alpha_k}$).

3. If $m = 2, N = 3$ and $\alpha_3 > \alpha_1 + \alpha_2$ (i.e. $f(u) = u_1^{\alpha_1} u_2^{\alpha_2} - u_3^{\alpha_3}$), then again global existence of classical solutions is proved in [12]. But the same result is not known if $\alpha_3 \leq \alpha_1 + \alpha_3$.

Besides those just mentioned, no more result of global classical solutions is proved to be valid for any space dimension n and any $d_k \in (0, \infty)$.

4. If again $m = 2, N = 4$, $(f(u) = u_1u_2 - u_3u_2)$, then global so-called *weak solutions* are proved to exist (see [13, 8]). *Weak solution* means that $f(u) \in L^1([0,T] \times \Omega))$ for all $T > 0$ and equations (2) are satisfied in the sense of distributions or in the sense of semigroups (see [13, 8, 14] for precise definitions).

5. More generally, if for some reason, the nonlinearity $f(u)$ is a priori bounded in $L^1((0,T) \times \Omega)$ for all $T > 0$, then global *weak solutions* do exist (see [13, 14]). Thanks to quadratic a priori estimates valid for these systems, this is for instance the case if

$$\begin{array}{l} N = m+1,\ f(u) = \prod_{k=1}^m u_k^{\alpha_k} - u_N^2; \\ N = m+2,\ f(u) = \prod_{k=1}^m u_k^{\alpha_k} - u_{m+1}u_{m+2} \end{array}$$

6. In the general situation of System (2), existence of global *weak solutions* in the above sense seems to be an open problem. No counterexample is known either. On the other hand, global existence of *still weaker solutions* is proved in [10]. They are called *renormalized solutions* and defined in the spirit of the famous renormalized solutions by Di Perna-Lions for the Boltzmann equation. A definition of such a solution for systems like (2) is introduced in [10] and global such solutions are also proved to exist in this same paper [10].

   We will not need the definition of such *renormalized solutions* here. We will only use the fact that they are obtained as limit of solutions of a standard approximate "regularized" system. And we will directly prove that any such limits are exponentially asymptotically stable. It is actually interesting to describe precisely the asymptotic behavior of these solutions without knowing much about them.

Let us consider the approximate solution $u^\varepsilon = (u_k^\varepsilon(x,t))$ to

$$\left\{ \begin{array}{l} \tau_k \frac{\partial u_k^\varepsilon}{\partial t} - d_k \Delta u_k^\varepsilon = \chi_k f_\varepsilon(u^\varepsilon) \quad \text{in } Q_T = \Omega \times (0,T) \\ \frac{\partial u_k^\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u_k^\varepsilon|_{t=0} = u_{k0}^\varepsilon(x) \geq 0, \quad 1 \leq k \leq N \end{array} \right. \tag{4}$$

where $\tau_k \in (0, \infty), 1 \leq k \leq N$ and

$$f_\varepsilon(u) = \frac{f(u)}{1 + \varepsilon|f(u)|}, \quad u_{k0}^\varepsilon = \inf\{u_{k0}, \varepsilon^{-1}\},\ u_{k0} \geq 0, 1 \leq k \leq N. \tag{5}$$

The introduction of the $\tau_k \neq 1$ is for later purposes (see Section 2.3). Note that $|f_\epsilon(u)| \leq 1/\epsilon$. Thus, given $(u_{k0}) \in L^1(\Omega)^N$, there exists a unique classical solution to (4)-(5) globally in time. Thanks to the *quasipositivity* of the nonlinearity, that is

$$\chi_k f_\epsilon(u) \geq 0,\ for\ all\ u \in [0,\infty)^N\ with\ u_k = 0,\ 1 \leq k \leq N,$$

this solution u^ϵ is nonnegative. Then, the following convergence result holds.

Proposition 1 *[10] Assume $u_{k0}\log u_{k0} \in L^1(\Omega)$ for $1 \le k \le N$. Then each $\{u^{\varepsilon_\ell}\}$ with $\varepsilon_\ell \downarrow 0$ admits a subsequence converging in $L^1_{loc}([0,\infty); L^1(\Omega)^N)$ and a.e. to some $u \in L^\infty([0,\infty); L^1(\Omega))^N$ such that*

$$u_k \log u_k \in L^\infty_{loc}([0,\infty); L^1(\Omega)) \text{ for all } 1 \le k \le N.$$

Remark 2 This proposition is essentially proved in [10]. We will give the needed extra details at the beginning of next section. When $\tau_k = 1$ for all k, the limit u is a *weak solution* of System (2), in the sense defined in the point 4 above, as soon as $f(u) \in L^1_{loc}([0,\infty); L^1(\Omega))$ (see [10] again). It is only a *renormalized solution* in the sense of [10] in general.

The conservation properties (where $⨍_\Omega$ denotes the average $|\Omega|^{-1}\int_\Omega$)

$$⨍_\Omega \tau_i u_i^\epsilon(t) + \tau_j u_j^\epsilon(t) = ⨍_\Omega \tau_i u_{i0}^\epsilon + \tau_j u_{j0}^\epsilon \text{ for all } 1 \le i \le m < j \le N, \tag{6}$$

hold, thanks to the homogeneous Neumann boundary conditions and they are preserved at the limit for u, at least a.e. $t \in [0,\infty)$. For $w : \Omega \to \mathbb{R}$, we will throughout denote

$$\overline{w} := ⨍_\Omega w.$$

Now the main result of this paper is the following theorem.

Theorem 3 *Let u be as in Proposition 1. Assume moreover that*

$$\overline{u}_{i0} + \overline{u}_{j0} > 0 \text{ for all } 1 \le i \le m < j \le N. \tag{7}$$

Then, there exists $C, a > 0$ depending only on $\|u_0\|_{L^1(\Omega)^N}$ and the data such that

$$\|u(\cdot,t) - z\|_{L^1(\Omega)^N} \le C e^{-a\,t}, \quad \forall t \ge 0 \tag{8}$$

where $z = (z_j)_{1 \le j \le N} \in (0,\infty)^N$ is the unique nonnegative solution of

$$f(z) = 0,\ \tau_i z_i + \tau_j z_j = \tau_i \overline{u}_{i0} + \tau_j \overline{u}_{j0} \text{ for all } 1 \le i \le m < j \le N. \tag{9}$$

The same conclusion would actually hold for *any limit* u of adequate approximate solutions of System (2), and not only for the solutions of the specific system (4), (5): this is discussed later in Remark 9.

The positivity condition (7) is not restrictive as explained in Section 5.

The asymptotic result of Theorem 3 has already been proved in the two particular situations of the points 3 et 4 above for 3×3 or 4×4 specific systems (see [5, 6, 7, 9]). As in these papers, the proof is based here on the use of the *entropy functional* defined as follows. Let

$$E(w \mid v) = ⨍_\Omega v\ \Phi\left(\frac{w}{v}\right)\ dx, \quad \Phi(s) = s(\log s - 1) + 1 \ge 0,\ \forall\, s > 0, \tag{10}$$

where w, v are measurable nonnegative functions (with $v(x)^2 + w^2(x) > 0$ a.e. $x \in \Omega$). This entropy is extended to the vector valued functions $u = (u_k)_{1\le k\le N}, z = (z_k)_{1\le N}$ as

$$\mathbf{E}(u \mid z) = \sum_{k=1}^{N} \tau_k E(u_k \mid z_k). \tag{11}$$

We will more simply write

$$E(w \mid 1) = E(w),\ \mathbf{E}(u) = \sum_{k=1}^{N} \tau_k E(u_k),\ \mathbf{E}(z) = \sum_{k} \tau_k E(z_k). \tag{12}$$

The main point is to prove that

Proposition 4 *With the notation and assumptions of Theorem 3*

$$\frac{d}{dt}\mathbf{E}(u(t) \mid z) \le -2a\, \mathbf{E}(u(t) \mid z), \tag{13}$$

in the sense of distributions in $(0, \infty)$.

By Proposition 1, $\mathbf{E}(u(t) \mid z)$ is bounded for t near 0 (say by C_0). Therefore (13) implies

$$\mathbf{E}(u(t) \mid z) \le C_0\, e^{-2a\,t},\ \forall\, t \ge 0. \tag{14}$$

We then apply a *Cziszár-Kullback type inequality*, namely (see Lemma 10)

$$\|u(t) - z\|_{L^1(\Omega)^N} \le C\, \mathbf{E}(u(t) \mid z),$$

which implies our main result (8).

Let us now recall the strategy to prove the main inequality (13). Assume for simplicity that, in the definition (3) of f and χ_k, we have

$$c_1 = c_2 = 1 = \alpha_k,\ \forall\, 1 \le k \le N. \tag{15}$$

Actually, we will see later that there is no loss of generality when considering this specific case (see Section 2.3). Then, if u is a solution of (2), we have, at least *formally*

$$\frac{d}{dt}E(u_k(t)) = \fint_\Omega \log u_k \partial_t u_k = \fint_\Omega -d_k \frac{|\nabla u_k|^2}{u_k} + \chi_k \log u_k f(u).$$

This implies that for $\mathbf{E}(u) = \sum_{k=1}^{N} E(u_k)$ (since here $\tau_k = 1$ for all k)

$$\frac{d}{dt}\mathbf{E}(u(t)) = -D(u(t)), \tag{16}$$

where

$$D(u) = 4\sum_{k=1}^N d_k \fint_\Omega |\nabla\sqrt{u_k}|^2 \\ + \fint_\Omega \left(\log\prod_{k=1}^m u_k - \log\prod_{k=m+1}^N u_k\right)\left(\prod_{k=1}^m u_k - \prod_{k=m+1}^N u_k\right). \tag{17}$$

Thanks to the definition of z, as proved in Lemma 7,

$$\mathbf{E}(u(t) \mid z) = \mathbf{E}(u(t)) - \mathbf{E}(z) \text{ so that } \frac{d}{dt}\mathbf{E}(u(t) \mid z) = \frac{d}{dt}\mathbf{E}(u(t)). \tag{18}$$

Now, Proposition 4 will be a consequence of the following lemma.

Lemma 5 *Assume (15). With the notation and assumptions of Theorem 3, the following holds*

$$D(u(t)) \geq 2a\,\mathbf{E}(u(t)|z), \tag{19}$$

in the sense of distribution on $(0,\infty)$.

It is now clear that combining (16), (18) and (19) yields Proposition 4, at least under Assumption (15) (and this will be general).

We prove in Section 2.3 why working with the particular case (15) is sufficient. The derivation in (16) is indeed very formal since here u is only obtained as the limit of regular solutions but may not be regular itself. In fact, we will only prove the inequality $\frac{d}{dt}\mathbf{E}(u(t)) \leq -D(u(t))$ which, obviously, is sufficient to deduce inequality (13) in Proposition 4. This will be done in Section 3 where a complete proof of Proposition 4 (and therefore of our main result of Theorem 3) will be given, assuming Lemma 5.

The proof of Lemma 5 is completely algebraic. It only uses from the solution $u(t)$ that is satisfies the conservation properties

$$\overline{u}_i(t) + \overline{u}_j(t) = \overline{u}_{i0} + \overline{u}_{j0} =: U_{ij},\ \forall\ 1 \leq i \leq m < j \leq N. \tag{20}$$

In the particular cases already known (namely in the points 3 and 4 above [5, 6, 7, 9]), this part of the proof is rather involved and requires much technicality. A main contribution here is to simplify rather significantly this part of the proof and consequently to be able to reach the general case (2). For instance, we compare the variation of $\sqrt{u}$ with the square root $\sqrt{\overline{u}}$ of its average rather than with the average of the square root. The corresponding computation turns out to be quite simpler and sufficient for the expected estimate of Lemma 13. We also simplify the proof of the estimate from below of $f(\sqrt{\overline{u}})$ (see Lemma 12)).

2 Some preliminaries

Let us first give the necessary extra details for the proof of Proposition 1.

2.1 Proof of Proposition 1.

Let us check that the results of [10] do apply here. Let us denote $U_k^\epsilon := \tau_k u_k^\epsilon$. Then System (4) may be rewritten

$$\begin{cases} \frac{\partial U_k^\varepsilon}{\partial t} - \frac{d_k}{\tau_k}\Delta U_k^\varepsilon = \chi_k \frac{F(U^\epsilon)}{1+\epsilon|F(U^\epsilon)|} \quad \text{in } Q_T = \Omega\times(0,T) \\ \frac{\partial U_k^\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad U_k^\varepsilon|_{t=0} = \tau_k u_{k0}^\varepsilon(x) \geq 0, \quad 1\leq k\leq N, \end{cases} \tag{21}$$

where, for all $U\in\mathbb{R}^N$

$$F(U) = C_1\prod_{i=1}^m U_i^{\alpha_i} - C_2\prod_{j=m+1}^N U_j^{\alpha_j},$$

$$C_1 = c_1\prod_{i=1}^m(\tau_i)^{-\alpha_i}, C_2 = c_2\prod_{j=m+1}^N(\tau_j)^{-\alpha_j}.$$

For this new system, the *entropy inequality* required in [10] holds, namely

$$\sum_{k=1}^N \chi_k F(U)[\mu_k + \log U_k] = -F(U)\left[\log\left(C_1\prod_{i=1}^m U_i^{\alpha_i}\right) - \log\left(C_2\prod_{j=m+1}^N U_j^{\alpha_j}\right)\right] \leq 0,$$

with $\mu_k = \log(C_2/C_1)/(N\chi_k), 1\leq k\leq N$. The a.e. convergence of U^ϵ (up to a subsequence) is stated in Lemma 7 of [10]. It implies the a.e. convergence of u^ϵ. Together with the estimate of $U_k^\epsilon \log U_k^\epsilon$ in $L^\infty_{loc}([0,\infty);L^1(\Omega))$, it also implies the convergence of U_k^ϵ and therefore of u_k^ϵ in $L^1_{loc}([0,\infty);L^1(\Omega))$. Morever, this implies that

$$u\in L^\infty([0,\infty);L^1(\Omega)) \text{ and } u_k\log u_k \in L^\infty_{loc}([0,\infty);L^1(\Omega)),\ \forall\, k.$$

□

2.2 Uniqueness of z.

We now prove the uniquenesse of z as defined in Theorem 3.

Proposition 6 *Under the assumptions of Theorem 3, there exists a unique* $z = (z_k)\in[0,\infty)^N$ *such that*

$$f(z) = 0, \quad \tau_i z_i + \tau_j z_j = \tau_i\overline{u}_{i0} + \tau_j\overline{u}_{j0},\ 1\leq i\leq m<j\leq N. \tag{22}$$

Moreover, $z_k > 0$, $\forall\, 1\leq k\leq N$.

Proof. Let $U_{ij} := \tau_i\overline{u}_{i0} + \tau_j\overline{u}_{j0}$. By (15), $U_{ij} > 0$ for $1\leq i\leq m<j\leq N$. The relations (22) are equivalent to

$$\begin{cases} z_j = [U_{1j} - \tau_1 z_1]/\tau_j \geq 0,\ \forall\, m+1\leq j\leq N, \\ z_i = [\tau_1 z_1 + U_{iN} - U_{1N}]/\tau_i \geq 0,\ \forall\, 2\leq i\leq m, \\ g(z_1) = 0, \end{cases} \tag{23}$$

where

$$g(z_1) := c_1 z_1 \prod_{i=2}^{m} \frac{[\tau_1 z_1 + U_{iN} - U_{1N}]^{\alpha_i}}{\tau_i^{\alpha_i}} - c_2 \prod_{j=m+1}^{N} \frac{[U_{1j} - \tau_1 z_1]^{\alpha_j}}{\tau_j^{\alpha_j}}.$$

Let us define

$$M_0 := \min_{m+1 \le j \le N} U_{1j}/\tau_1, \;\; m_0 := \max_{2 \le i \le m} [U_{1N} - U_{iN}]^+/\tau_1.$$

Note that $U_{1N} - U_{iN} = U_{1j} - U_{ij} = \tau_1 \overline{u}_{10} - \tau_i \overline{u}_{i_0}$ is independent of $j = m+1, ..., N$. It follows that $m_0 < M_0$. The function $g : [m_0, M_0] \to \mathbb{R}$ is continuous, strictly increasing and satisfies $g(m_0) < 0,\ g(M_0) > 0$. Therefore there exists a *unique* $z_1 \in (m_0, M_0)$ such that $g(z_1) = 0$. For this z_1, the z_i, z_j defined by (23) are nonnegative and do satisfy the expected relations (22). They are all stricly positive: indeed, if one had $z_i = 0$ for some $1 \le i \le m$, then $f(z) = 0$ would imply that $z_j = 0$ also for some $m+1 \le j \le N$ which is a contradiction with $\tau_i z_i + \tau_j z_j = U_{ij} > 0$. □

2.3 Reduction of System (4) to the case $c_1 = c_2 = 1, \alpha_k = 1, 1 \le k \le N$.

Let us show that we may only consider these particular values without loss of generality. Let us check that System (4) is actually a particular case of the next System (24) whose solutions are exactly α_k copies of $u_k, 1 \le k \le N$. Let us define

$$l_0 = 0, l_k = \sum_{j=1}^{k} \alpha_j,\ \forall\, 1 \le k \le N;\ \lambda^{-l_m} := c_1,\ \mu^{l_m - l_N} := c_2,$$

$$D^l := \lambda d_k/\alpha_k,\ \tau^l := \lambda \tau_k/\alpha_k,\ \forall\ l_{k-1} < l \le l_k, \forall\, 1 \le k \le m,$$
$$D^l := \mu d_k/\alpha_k,\ \tau^l := \mu \tau_k/\alpha_k,\ \forall\ l_{k-1} < l \le l_k, \forall\, m+1 \le k \le N.$$

And we consider the extended system

$$\left\{ \begin{array}{l} \tau^l \frac{\partial v^{l,\epsilon}}{\partial t} - D^l \Delta v^{l,\epsilon} = \chi^l g(v^\epsilon)/[1 + \epsilon\, |g(v^\epsilon)|] \quad \text{in } Q_T = \Omega \times (0,T), \\ \frac{\partial v^l}{\partial \nu}\Big|_{\partial\Omega} = 0,\ g(v^\epsilon) = \prod_{l=1}^{l_m} v^{l,\epsilon} - \prod_{l=l_m+1}^{l_N} v^{l,\epsilon},\ \ v^\epsilon = (v^{l,\epsilon})_{1 \le l \le l_N}, \\ \chi^l = -1,\ v^l|_{t=0} = u_{k0}/\lambda,\ \forall\, 1 \le l \le l_m, \\ \chi^l = 1,\ v^l|_{t=0} = u_{k0}/\mu,\ \forall\, l_m < l \le l_N. \end{array} \right. \tag{24}$$

By uniqueness, we have

$$v^{l,\epsilon} = v^{l_k,\epsilon},\ \forall\, l_{k-1} < l \le l_k,\ 1 \le k \le N.$$

Let us set

$$u_k^\epsilon := \lambda v^{l_k,\epsilon},\ \forall\, 1 \le k \le m,\ u_k := \mu v^{l_k,\epsilon},\ \forall\, m+1 \le k \le N.$$

Then, we check that u_k^ϵ is the solution of System (4). □

We will now always assume that

$$c_1 = c_2 = 1, \ \ \alpha_k = 1, \ \forall\, 1 \le k \le N, \tag{25}$$

3 Lemma 5 implies Theorem 3

Let us first note the following identity.

Lemma 7 *Under the assumptions of Lemma 5*

$$\mathbf{E}(u(t)|z) = \mathbf{E}(u(t)) - \mathbf{E}(z), \ \forall\, t \ge 0. \tag{26}$$

Proof. The function $E(\cdot \mid \cdot), \mathbf{E}(\cdot \mid \cdot), E(\cdot), \mathbf{E}(\cdot)$ are defined in (10), (11), (12). The following property is valid for any $w \in L^1(\Omega)^+$ and $w_* \in (0, \infty)$:

$$E(w \mid w_*) = E(w) - E(w_*) - (\overline{w} - w_*) \log w_*. \tag{27}$$

We apply this to $w = u_k(t), w_* = z_k$ for all $1 \le k \le N$ and we sum over k. Then (26) is reduced to checking

$$\sum_{k=1}^{N} \tau_k (\overline{u}_k(t) - z_k) \log z_k = 0. \tag{28}$$

We have by (6) and for all $\epsilon > 0$

$$\tau_i \overline{u}_i^\epsilon(t) + \tau_j \overline{u}_j^\epsilon(t) = \tau_i \overline{u}_{i0}^\epsilon + \tau_j \overline{u}_{j0}^\epsilon, \ \forall 1 \le i \le m < j \le N.$$

This is preserved at the limit and gives

$$\tau_i \overline{u}_i(t) + \tau_j \overline{u}_j(t) = \tau_i \overline{u}_{i0} + \tau_j \overline{u}_{j0}, \ \forall 1 \le i \le m < j \le N. \tag{29}$$

Since $\tau_i z_i + \tau_j z_j = \tau_i \overline{u}_{i0} + \tau_j \overline{u}_{j0}$, this may be rewritten as

$$\tau_k (\overline{u}_k(t) - z_k) = \begin{cases} \tau_1 (\overline{u}_1(t) - z_1) \ \textit{for all}\ 1 \le k \le m, \\ -\tau_1 (\overline{u}_1(t) - z_1) \ \textit{for all}\ m+1 \le k \le N, \end{cases} \tag{30}$$

Then we write (28) as

$$\sum_{k=1}^{N} \tau_k (\overline{u}_k(t) - z_k) \log z_k = \tau_1 (\overline{u}_1(t) - z_1) \left\{ \sum_{k=1}^{m} \log z_k - \sum_{k=m+1}^{N} \log z_k \right\} = 0,$$

using $f(z) = 0$ (recall that (25) holds so that $f(z) = \prod_{i=1}^m z_i - \prod_{j=m+1}^N z_j$). $\square$

We now show the key lemma of this section.

Lemma 8 *With the notation and assumptions of Theorem 3, together with (25), we have*

$$\frac{d}{dt}\mathbf{E}(u) \leq -D(u) \tag{31}$$

in the sense of distributions on $(0, \infty)$.

Proof. For the classical solution $u^\epsilon = (u_k^\epsilon(\cdot, t))$ to approximate scheme (4)-(5), it holds that

$$\frac{d}{dt}\mathbf{E}(u^\epsilon) + D_\epsilon(u^\epsilon) = 0, \tag{32}$$

where (together with (25))

$$D_\epsilon(u) = 4\sum_{k=1}^{N} d_k \|\nabla\sqrt{u_k}\|_2^2 + \int_\Omega \frac{f(u)}{1+\epsilon|f(u)|} \log\frac{\prod_{k=1}^m u_k}{\prod_{k=m+1}^N u_k} \geq 0. \tag{33}$$

Inequality (32) implies after integration in time

$$\mathbf{E}(u^\varepsilon(\cdot, t)) \leq \mathbf{E}(u_0^\varepsilon), \iint_{Q_T} |\nabla\sqrt{u_k^\varepsilon}|^2 \leq C, \quad 1 \leq k \leq N. \tag{34}$$

From the first inequality in (34), using Proposition 1 and Fatou's lemma, we deduce

$$\mathbf{E}(u(\cdot, t)) \leq \mathbf{E}(u_0) \quad \text{a.e. } t. \tag{35}$$

Let us prove that, up to a subsequence,

$$\lim_{\ell\to\infty} \mathbf{E}(u^{\varepsilon_\ell}(\cdot, t)) = \mathbf{E}(u(\cdot, t)) \quad \text{a.e. } t \in (0, \infty), \tag{36}$$

We have

$$\frac{\partial}{\partial t}(\tau_i u_i^\varepsilon + \tau_j u_j^\epsilon) - \Delta(d_i u_i^\varepsilon + d_j u_j^\varepsilon) = 0 \quad \text{in } Q_T$$
$$\frac{\partial}{\partial \nu}(d_i u_i^\varepsilon + d_j u_j^\varepsilon)\Big|_{\partial\Omega} = 0, \quad u_k^\varepsilon|_{t=0} = u_{k0}^\varepsilon$$

for $1 \leq i \leq m < j \leq N$, and $1 \leq k \leq N$. Then Lemma 4 of [15] implies

$$\|u^\varepsilon\|_{L^2(Q_{\tau,T})} \leq C_{\tau,T} \tag{37}$$

for any $\tau \in (0, T)$ with $C_{\tau,T} > 0$ independent of ε, where $Q_{\tau,T} = \Omega \times (\tau, T)$. (See Proposition 6.1 of [14] when $(u_{k0}) \in L^2(\Omega)^N$ in which case we may take $\tau = 0$). Since u^{ϵ_l} tends to u a.e. (see Proposition 1), we classically deduce (36) from Egorov's theorem and the estimate (37). Indeed, given $\alpha > 0$, there exists a compact set $K_\alpha \subset Q_{\tau,T}$ such that $u^{\varepsilon_\ell} \to u$ uniformly on K_α and $|Q_{\tau,T} \setminus K_\alpha| < \alpha$. With $\Phi(s) = s[\log s - 1] + 1$ as in (10), since for some $C \in (0, \infty)$

$$0 \leq \Phi(s)^{3/2} \leq C(s^2 + 1), \quad s > 0,$$

it holds by (37) that

$$\iint_{Q_{\tau,T}\setminus K_\alpha} |\Phi(u^\varepsilon)-\Phi(u)|\ dxdt \le |Q_{\tau,T}\setminus K_\alpha|^{1/3} \cdot \left(\iint_{Q_{\tau,T}\setminus K_\alpha} |\Phi(u^\varepsilon)-\Phi(u)|^{3/2} \right)^{2/3} \le C\alpha^{1/3}.$$

Hence

$$\limsup_{\ell\to\infty} \iint_{Q_{\tau,T}} |\Phi(u^{\varepsilon_\ell})-\Phi(u)|\ dxdt \le C\alpha^{1/3}.$$

Letting $\alpha \downarrow 0$, we obtain (recall the definition of $\mathbf{E}$ in (11), 12))

$$\lim_{\ell\to\infty} \int_\tau^T |\mathbf{E}(u^{\varepsilon_l}(\cdot,t)) - \mathbf{E}(u(\cdot,t))|\ dt = 0,$$

and therefore (36) passing to a subsequence.

Let $\phi \in C_0^\infty[0,T)^+$. It holds that

$$\phi(0)\mathbf{E}(u_0^\varepsilon) + \int_0^\infty \phi'(t)\mathbf{E}(u^\varepsilon(\cdot,t))\ dt = \int_0^\infty \phi(t) D_\varepsilon(u^\varepsilon(\cdot,t))\ dt \tag{38}$$

by (32). As $\varepsilon = \varepsilon^\ell \downarrow 0$, the left-hand side of (38) converges to

$$\phi(0)\mathbf{E}(u_0) + \int_0^\infty \phi'(t)\mathbf{E}(u(\cdot,t))\ dt.$$

Here, we used the dominated convergence theorem, recalling (36) with (35) and $(u_{k0}\log u_{k0}) \in L^1(\Omega)^N$.

To treat the right-hand side of (38), we recall the expression of $D_\epsilon(u^\epsilon)$ in (33). For its first term, we use (34) to deduce the weak convergence,

$$\nabla\sqrt{u_k^{\varepsilon_\ell}} \rightharpoonup \nabla\sqrt{u_k} \quad \text{in } L^2(Q_T)^N \text{ for } 1 \le k \le N,$$

passing to a subsequence. Fatou's lemma is applicable to the second term and it follows that

$$\liminf_{\ell\to\infty} \int_0^\infty \phi(t) D_{\varepsilon_\ell}(u^{\varepsilon_\ell}(\cdot,t))\ dt \ge \int_0^\infty \phi(t) D(u(\cdot,t))\ dt.$$

We thus end up with

$$\phi(0)\mathbf{E}(u_0) + \int_0^\infty \phi'(t)\mathbf{E}(u(\cdot,t))\ dt \ge \int_0^\infty \phi(t) D(u(\cdot,t))\ dt$$

which means (31) on $[0,\infty)$ in the sense of distributions, because $T>0$ and $\phi \in C_0^\infty[0,\infty)^+$ are arbitrary. □

Remark 9 Analyzing the above proof shows that the same result would hold for quite more general approximations f_ϵ of f. For instance, we could choose

$$f_\epsilon(s) = f(s)G_\epsilon(s),\ 0 \le G_\epsilon(s) \le M,\ |f_\epsilon(s)| \le 1/\epsilon,\ \textit{for all}\ s \in [0,\infty)^N,$$

with $f_\epsilon(s) \to f(s)$ as $\epsilon \to 0^+$. Then any pointwise limit of the corresponding approximate solution would satisfy the conclusion of Lemma 8 and of Theorem 3 as well.

The following lemma is an adaptation of the classical Cziszár-Kullback inequality to our situation in the spirit of [5, 6, 7, 9].

Lemma 10 *With the notation and assumptions of Theorem 3,*

$$\|u(t) - z\|_{L^1(\Omega)^N} \le C\,\mathbf{E}(u(t) \mid z),\ \forall\, t \ge 0,$$

for some $C > 0$ *depending on* u_0, z *and the data.*

Proof. For $\Phi(s) = s(\log s - 1) + s$ as defined by (10), we have

$$\forall\, s \in [0, M],\ |s-1|^2 \le C(M)\,\Phi(s).$$

We deduce

$$|\overline{u}_k(t) - z_k|^2 = z_k^2 \left|\frac{\overline{u}_k(t)}{z_k} - 1\right|^2 \le C\, z_k \Phi\left(\frac{\overline{u}_k(t)}{z_k}\right), \quad 1 \le k \le N,$$

where C depends only on $\|u_0\|_{L^1(\Omega)^N}, \|z\|$. It follows that, for some $C_1 > 0$

$$C_1[\|\overline{u}(t) - z\|_{L^1(\Omega)^N}]^2 \le \sum_{k=1}^N \tau_k |\overline{u}_k(t) - z_k|^2 \le C\,\mathbf{E}(\overline{u}(t) \mid z). \tag{39}$$

Now the classical Cziszár-Kullback-Pinsker inequality says (see e.g. Theorem 31 in [3] or also [4])

$$\left[\fint_\Omega |u_k(t) - \overline{u}_k(t)|\right]^2 \le 4\overline{u}_k(t) E(u_k(t) \mid \overline{u}_k(t)).$$

This implies, for some other constant C

$$\|u(t) - \overline{u}(t)\|^2_{L^1(\Omega)^N} \le C\,\mathbf{E}(u(t) \mid \overline{u}(t)). \tag{40}$$

Using the obvious relation $\mathbf{E}(u(t) \mid z) = \mathbf{E}(u(t) \mid \overline{u}(t)) + \mathbf{E}(\overline{u}(t) \mid z)$ together with (39) and (40), we obtain with another constant C

$$\|u(t) - z\|^2_{L^1(\Omega)^N} \le C\,\mathbf{E}(u(t) \mid z),$$

which is the estimate of Lemma 10. $\square$

Proof of Theorem 3. As proved in Section 2.3, we may assume (25). By Lemmas 8, 7 and 5, we obtain

$$\frac{d}{dt}\mathbf{E}(u \mid z) \le -2a\,\mathbf{E}(u \mid z)$$

in the sense of distributions on $(0,\infty)$. This is the statement of Proposition 4 and it implies

$$\mathbf{E}(u(\cdot,t) \mid z) \le Ce^{-2at}, \quad t \ge 0. \tag{41}$$

Together with Lemma 10, this implies Theorem 3. □

4 Proof of Lemma 5

This proof is inspired from those given in [5, 6, 7, 9] for the 4×4 systems, with some significant improvements and simplifying modifications as explained in the introduction.

Here we denote by u_k, u any of the functions $u_k(t), u(t)$ without indicating the t dependence (which is actually not used in this section). Only the conservation laws (see (29))

$$\tau_i\overline{u}_i^k + \tau_j\overline{u}_j^k = U_{ij} := \tau_i\overline{u}_{i0} + \tau_j\overline{u}_{j0},\ \forall 1 \le i \le m < j \le N,$$

will be used together with the simplified assumption (25) and the following properties

$$0 < U_0 := \min_{i,j} U_{ij},\ \ \max_{i,j} U_{i;j} \le (\sum_{k=1}^{N}\tau_k)\|u_0\|_{L^1(\Omega)^N},\ 0 < \tau_0 = \min_k \tau_k. \tag{42}$$

All constants $'C'$ below will depend only on $U_0, \|u_0\|_{L^1(\Omega)^N}, \tau_k, 1 \le k \le N$.

Lemma 11 *It holds that*

$$\mathbf{E}(\overline{u} \mid z) \le C\sum_{k=1}^{N}\left(\sqrt{\overline{u}_k} - \sqrt{z_k}\right)^2. \tag{43}$$

Proof. It is easily seen that $B(s) := \Phi(s)/(\sqrt{s}-1)^2$ is continuous on $[0,\infty)$. Thus $B(\overline{u}_k/z_k)$ is bounded above by the constants in (42). And we have

$$\mathbf{E}(\overline{u} \mid z) = \sum_{k=1}^{N}\tau_k z_k \Phi\left(\frac{\overline{u}_k}{z_k}\right) = \sum_k \tau_k z_k\left(\frac{\sqrt{\overline{u}}_k}{\sqrt{z}_k} - 1\right)^2 B\left(\frac{\overline{u}_k}{z_k}\right).$$

$$\le C\sum_k(\sqrt{\overline{u}}_k - \sqrt{z}_k)^2,$$

whence Lemma 11. □

Lemma 12 *It holds that*

$$\sum_{k=1}^{N}(\sqrt{\overline{u}_k}-\sqrt{z_k})^2 \le C\,\left[f(\sqrt{\overline{u}})\right]^2, \quad \sqrt{\overline{u}}=(\sqrt{\overline{u}_k})_{1\le k\le N}. \tag{44}$$

Proof. Recall that, under the assumption (25), $f(u)=\prod_{i=1}^m u_i-\prod_{j=m+1}^N u_j$. According to (30), we have

$$\left\{\begin{array}{l} \overline{u}-z=\theta e, \quad \theta=\overline{u}_1-z_1, \quad e=(e_k)_{1\le k\le N}, \\ e_i=\tau_1/\tau_i,\ e_j=-\tau_1/\tau_j,\ \forall\, 1\le i\le m<j\le N. \end{array}\right. \tag{45}$$

Therefore

$$f(\overline{u})=f(\overline{u})-f(z)=\left[\int_0^1 \nabla f((1-s)z+s\overline{u})\;ds\right]\cdot(\overline{u}-z)=L(\overline{u})(\overline{u}_1-z_1), \tag{46}$$

where

$$L(\zeta)=\int_0^1 \nabla f((1-s)z+s\zeta)\cdot e\;ds, \quad 0\le\zeta\in\mathbb{R}^N. \tag{47}$$

We have $\overline{u}=z+(\overline{u}_1-z_1)\,e$ where $\overline{u}_1\in I:=[0,\min_{m<j\le N}U_{1j}]$. But the mapping $\sigma\in I\mapsto L(z+(\sigma-z_1)\,e)$ is continuous. It does not vanish: indeed, if one had $L(\zeta)=0$ for some $\zeta=z+(\sigma-z_1)e,\sigma\in I$, then, by the same computation as in (46) with $\overline{u}$ replaced by ζ, we would also have $f(\zeta)=0$. But the uniqueness property of Proposition 6 would imply $\zeta=z$. And this is impossible since then $L(z)=0$ and by (47),

$$L(z)=\nabla f(z)\cdot e=\tau_1\left[\sum_{i=1}^m(\tau_i z_i)^{-1}\prod_{k=1}^m z_k+\sum_{j=m+1}^N(\tau_j z_j]^{-1}\prod_{k=m+1}^N z_k\right],$$

whence a contradiction.

Thus, for

$$\delta=\min_{\sigma\in I}L(z+(\sigma-z_1)e)>0,$$

it holds that $L(\overline{u})\ge\delta$, which implies by (46) and (45)

$$f(\overline{u})^2=(L(\overline{u}))^2(\overline{u}_1-z_1)^2\ge\delta^2\|\overline{u}-z\|^2/\|e\|^2,$$

where $\|\cdot\|$ denotes here the euclidean norm in $\mathbb{R}^N$. We combine this with the identities

$$\begin{array}{rcl} (\overline{u}_k-z_k)^2 & = & (\sqrt{\overline{u}_k}-\sqrt{z_k})^2(\sqrt{\overline{u}_k}+\sqrt{z_k})^2 \\ & \ge & \left(\min\limits_{1\le k\le N} z_k\right)\cdot(\sqrt{\overline{u}_k}-\sqrt{z_k})^2, \quad 1\le k\le N \end{array}$$

and with

$$\begin{array}{rcl} f(\overline{u})^2 & = & \left(\prod\limits_{i=1}^m\overline{u}_i-\prod\limits_{j=m+1}^N\overline{u}_j\right)^2=f(\sqrt{\overline{u}})^2\cdot\left(\prod\limits_{i=1}^m\sqrt{\overline{u}_i}+\prod\limits_{j=m+1}^N\sqrt{\overline{u}_j}\right)^2 \\ & \le & Cf(\sqrt{\overline{u}}\,)^2 \end{array}$$

to deduce (44). $\square$

Lemma 13 *It holds that*

$$\left[f(\sqrt{\bar{u}})\right]^2 \le C\fint_\Omega f(\sqrt{u})^2 + \sum_k |\nabla\sqrt{u_k}|^2 \tag{48}$$

for $\sqrt{u} = (\sqrt{u_k}\,)_{1\le k\le N}$.

Proof. All constant C in this proof may again differ from each other but will depend only on the value in (42). Define $\sigma = \sigma(x) \in \mathbb{R}^N$ for $x \in \Omega$ by $\sqrt{u} = \sqrt{\bar{u}} + \sigma$. First, we have

$$f(\sqrt{u})^2 = f(\sqrt{\bar{u}}+\sigma)^2 = \Big(f(\sqrt{\bar{u}}) + \nabla f(\sqrt{\bar{u}})\cdot\sigma + M\Big)^2,$$

where $M = \int_0^1 (1-s)D^2 f(\sqrt{\bar{u}}+s\sigma)[\sigma,\sigma]\,ds$. Using $(\nabla f(\sqrt{\bar{u}})\cdot\sigma + M)^2 \ge 0$, this implies

$$f(\sqrt{u})^2 \ge f(\sqrt{\bar{u}})^2 + 2f(\sqrt{\bar{u}})\nabla f(\sqrt{\bar{u}})\cdot\sigma + 2f(\sqrt{\bar{u}})M.$$

By Young's inequality and the estimate $|\nabla f(\sqrt{\bar{u}}\,)\cdot\sigma| \le C\|\sigma\|$, we have

$$2f(\sqrt{\bar{u}})\nabla f(\sqrt{\bar{u}})\cdot\sigma \ge -\frac{1}{2}f(\sqrt{\bar{u}})^2 - 2(\nabla f(\sqrt{\bar{u}})\cdot\sigma)^2 \ge -\frac{1}{2}f(\sqrt{\bar{u}})^2 - C\|\sigma\|^2.$$

It follows from the two previous inequalities and $|f(\sqrt{\bar{u}}\,)| \le C$ that

$$f(\sqrt{u})^2 \ge \frac{1}{2}f(\sqrt{\bar{u}})^2 - C(\|\sigma\|^2 + |M|). \tag{49}$$

Next, since $\sqrt{u} \ge 0$ implies $\sigma \ge -\sqrt{\bar{u}}$ in $\mathbb{R}^N$, we have the partition $\Omega = \Omega_1 \cup \Omega_2$ where

$$\Omega_1 = \{x \in \Omega \mid -\sqrt{\bar{u}_k} \le \sigma_k(x) \le 1,\ \forall\ 1 \le k \le N\},$$
$$\Omega_2 = \cup_{1\le k\le N}\{x \in \Omega \mid \sigma_k(x) > 1\}.$$

For $x \in \Omega_1, s \in [0,1]$, one has: $0 \le \sqrt{\bar{u}_k} + s\sigma_k \le 1 + \sqrt{\bar{u}}_k$, so that

$$|M| \le \int_0^1 (1-s)\|D^2 f(\sqrt{\bar{u}}+s\sigma)\|\ ds\cdot\|\sigma\|^2 \le C\|\sigma\|^2,\ x\in\Omega_1.$$

Together with (49), we deduce

$$\int_{\Omega_1} f(\sqrt{u})^2\ dx \ge \int_{\Omega_1}\left[\frac{1}{2}f(\sqrt{\bar{u}})^2 - C\|\sigma\|^2\right]dx. \tag{50}$$

We also have

$$\int_{\Omega_2} f(\sqrt{\bar{u}})^2\ dx = |\Omega_2| f(\sqrt{\bar{u}})^2 \le f(\sqrt{\bar{u}})^2\sum_{k=1}^N \big|\{\sigma_k^2 > 1\}\big|$$

with

$$\big|\{\sigma_k^2>1\}\big| = \int_{\{\sigma_k^2>1\}} dx \le \int_{\{\sigma_k^2>1\}}\sigma_k^2\ dx \le \int_\Omega \sigma_k^2\ dx,$$

which implies

$$\int_{\Omega_2} f(\sqrt{\overline{u}})^2 \; dx \leq f(\sqrt{\overline{u}})^2 \int_{\Omega} \|\sigma\|^2 \; dx \leq C \int_{\Omega} \|\sigma\|^2 \; dx. \tag{51}$$

By (50)-(51), we obtain

$$f(\sqrt{\overline{u}})^2 = \fint_{\Omega} f(\sqrt{\overline{u}})^2 \; dx \leq C \fint_{\Omega} [f(\sqrt{u})^2 + \|\sigma\|^2] \; dx. \tag{52}$$

Then, using in particular Schwarz inequality: $\sqrt{\overline{u}_k} \geq \fint_{\Omega} \sqrt{u_k}$, we have

$$\fint_{\Omega} \sigma_k^2 = \fint_{\Omega} u_k - 2\sqrt{\overline{u}_k}\sqrt{u_k} + \overline{u}_k \leq 2 \left\{ \fint u_k - \left(\fint_{\Omega} \sqrt{u_k} \right)^2 \right\} = 2 \fint_{\Omega} \left(\sqrt{u_k} - \fint_{\Omega} \sqrt{u_k} \right)^2.$$

Using now Poincaré-Wirtinger's inequality implies that

$$\fint_{\Omega} \sigma_k^2 = 2 \fint_{\Omega} \left(\sqrt{u_k} - \fint_{\Omega} \sqrt{u_k} \right)^2 \leq C \fint_{\Omega} |\nabla \sqrt{u_k}|^2.$$

Whence (48) by plugging this inequality for all $k = 1, ..., N$ into (52). □

Proof of Lemma 5. Combining Lemmas 11, 12, and 13, we obtain

$$\mathbf{E}(\overline{u} \mid z) \leq C \fint_{\Omega} f(\sqrt{u})^2 + \sum_k |\nabla \sqrt{u_k}|^2. \tag{53}$$

Here, the elementary inequality

$$\left(\sqrt{Y} - \sqrt{X} \right)^2 \leq (Y - X) \log \frac{Y}{X}, \quad X, Y \geq 0,$$

applied to $Y = \prod_{i=1}^m u_i$, $X = \prod_{j=m+1}^N u_j$, implies that

$$f(\sqrt{u})^2 \leq f(u) \left(\log \prod_{i=1}^m u_i - \log \prod_{j=m+1}^N u_j \right)$$

and hence

$$\fint_{\Omega} f(\sqrt{u})^2 \leq \fint_{\Omega} \left(\log \prod_{i=1}^m u_i - \log \prod_{j=m+1}^N u_j \right) \left(\prod_{i=1}^m u_i - \prod_{j=m+1}^N u_j \right).$$

From this inequality and (53), we obtain

$$\mathbf{E}(\overline{u} \mid z) \leq C D(u). \tag{54}$$

Finally, we use the additivity property $\mathbf{E}(u \mid z) = \mathbf{E}(u \mid \overline{u}) + \mathbf{E}(\overline{u}, z)$ and the logarithmic Sobolev inequality (see e.g. Theorem 17 in [3])

$$E(u_k, \overline{u}_k) \leq C \fint_{\Omega} |\nabla \sqrt{u_k}|^2, \; 1 \leq k \leq N,$$

to deduce the statement of Lemma 5. □

5 Concluding remarks

The main result of Theorem 3 is proved under the positivity assumption (7). This is actually not a restriction. Indeed, if one has $⨍_\Omega u_{i0} + u_{j0} = 0$ for some $1 \leq i \leq m < j \leq N$, in other words if $u_{i0} \equiv 0 \equiv u_{j0}$, then by uniqueness, $u_i^\epsilon(t) \equiv 0 \equiv u_j^\epsilon(t), f(u^\epsilon) \equiv 0$ and System (2) is reduced to the heat equation for each u_k. It is well known in this case that $u_k(t)$ converges exponentially as $t \to \infty$ to the average $⨍_\Omega u_{k0}$.

On the other hand, Theorem 3 does not handle the interesting case when the chemical species are not separated, contrary to the reversible reaction (1). This is the case for instance with the typical following reaction

$$A_1 + 2\,A_2 \rightleftharpoons 2\,A_1 + A_2.$$

The corresponding system writes

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1 u_2^2 - u_1^2 u_2 = -\left[\frac{\partial u_2}{\partial t} - d_2 \Delta u_2\right], \\ \frac{\partial u_1}{\partial \nu} = 0 = \frac{\partial u_2}{\partial \nu}, \quad (u,v)|_{t=0} = (u_0(x), v_0(x)) \geq 0. \end{cases} \tag{55}$$

Here, the only positive solution of System (22), namely of

$$z_1 z_2^2 = z_1^2 z_2,\ z_1 + z_2 = \overline{u}_{10} + \overline{u}_{20} := U_{12},$$

is given by $z = (U_{12}/2, U_{12}/2)$. But the situation is quite different from Theorem 3. Indeed if $U_{12} > 0$, the solution does not always converge to this z. If we chose for instance, $u_{10} \equiv 0, u_{20} \equiv a > 0$, then, by uniqueness, the solution is independent of the space variable x and is given by $(u_1(t), u_2(t)) = (0, a)$. Actually, the solution of the spatially homogeneous part of this system is given by $(u_1(t), u_2(t)) = (v(t), a - v(t))$ where v is solution of

$$v' = v(a - v)(a - 2v).$$

And this equation has three stationary states, 0, $m_0/2$, m_0. The second one is stable, while the first and the third ones are unstable. Such a behavior probably holds for System (55) and more generally, for systems corresponding to general reversible chemical reactions with all $A_1, ..., A_N$ appearing on both sides.

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